A GENERALIZATION OF SELBERG INTEGRAL

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Abstract. We analyze the situation which is related to zonal spherical functions of type A_n and obtain a generalization of Selberg integral.

0.1 Notations.

R -set of roots of root system of type A_n

R_+ - set of positive roots

$\alpha_1, \alpha_2, \dots, \alpha_n$ - simple roots of root system of type A_n

$\delta = \frac{1}{2}\sum_{\alpha\in R_+}\alpha$ -halfsum of positive roots

$\Lambda_1, \Lambda_2, \dots, \Lambda_n$ -fundamental weights, i.e.

$(\alpha_i, \Lambda_j) = \delta_{ij}$, where δ_{ij} is Kronecker's delta

k- complex parameter ('halfmultiplicity' of a root)

$$\rho = \rho(k) = \frac{k}{2}\sum_{\alpha\in R_+}\alpha$$

Let $\mathbb{R}^{n+1}$ be a $(n+1)$-dimensional Euclidean vector space with inner product $(.,.)$ and with basis $e_1, e_2, \dots e_{n+1}$. We realize simple roots $\alpha_1, \alpha_2, \dots, \alpha_n$ as $e_1 - e_2, e_2 - e_3, \dots, e_n - e_{n+1}$, correspondingly.

$\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}$

$W = S_{n+1}$ Weyl group of type A_n (group generated by the orthogonal reflections with respect to hyperlanes perpendicular to roots $\alpha \in R$)

$\{z_l,\ \ l = 1, \dots, n+1\}$ - arguments

$\{t_{ij},\ i = 1, \dots, j,\ \ j = 1, \dots, n\}$ - variables of integration

$$\lambda = (\lambda_1, \dots, \lambda_{n+1})\ |\lambda_1 + \lambda_2 + \dots + \lambda_{n+1} = 0$$

Though this homogeneity condition might be released we prefer to impose it. Also parameter λ is assumed to be generic.

$\phi(\lambda + \rho(k), k, z)$- asymptotic solution with the leading asymptotic $z^{\lambda+\rho}$, i.e.

$\phi(\lambda + \rho, k, z) = z^{\lambda+\rho}(1 + \dots)$

$\Delta_w(z)$, $w \in W$ - cycles for asymptotic solutions, cf. ref.[43]

1. Introduction

We analyeze the situation which is related to zonal spherical functions of type A_n. The situation is also known as Calogero-Sutherland model. The zonal spherical functions on symmetric Rimannian spaces were introduced in ref. [70]. In the case of root system of type A_n it is by proven by I.Cherednik and A.Matsuo in refs. [38,39] that hypergeometric system of differential equations of Heckman-Opdam cf. ref. [45] is related to the particular case of trigonometric version of Knizhnik-Zamolodchikov equation in conformal field theory. In particular, solutions to the hypergeometric system of Heckman-Opdam can be obtained from the solutions of Knizhnik-Zamoldchikov equations by symmetrization procedure. Solutions to Knizhnik-Zamolodchikov equations are given by certain multidimensional integrals, whose integrand has the standard part times complicated meromorphic function cf. refs. [40, 32]. This complicated meromorphic factor becomes even more complicated (formally) after symmetrization. We would like like to emphasize that in this particular case one can get rid of this unpleasent meromorphic factor cf. theorem 3.2. below. See also the refs. [16 ,68] , as well as refs. [19,15,69] about W-algebras.

Here is the organization of the paper. In section two we recall the transformation law for the Heckman-Opdam hypergeometric functions related to root system cf. ref.[48]. In section 3 using the integral representation in the case of root system of type A_n from ref. [43] with the help of transformation law we obtain another integral representation (theorem 3.2) and calculate the leading coefficient. In section 4 using the evaluation theorem of Opdam (theorem 4.4 below) we obtain generalized Selberg integral (theorem 4.1). Remarkably, the answer is the same (up to the phase) for $(n+1)!$ different contours of integration.

Recently integrals of this type have drawn much attention because of applications to conformal field theory, cf. ref. [2].

2. Transformation law

2.1. Differential operator of second order.

Let L be the following differential operator

$$L = L(k) = \sum_{i=1}^{n+1} (z_i \frac{\partial}{\partial z_i})^2 - k \sum_{i<j} \frac{z_j + z_i}{z_j - z_i} (z_i \frac{\partial}{\partial z_i} - z_j \frac{\partial}{\partial z_j}).$$

Remark 2.2. Operator L originates in the theory of zonal spherical functions as the radial part of Laplace-Casimir operator of second order taken with respect to Cartan decomposition $G = KAK$ cf. refs.[46,9].

2.3 Important property of operator L.

$$\prod(z_i-z_j)^{2k-1}\prod z_i^{(1-2k)\frac{n}{2}}\circ L(k)\circ\prod(z_i-z_j)^{1-2k}\prod z_i^{(2k-1)\frac{n}{2}}=L(1-k)+(1-2k)(\delta,\delta)$$

cf. Proposition 4.2. ref. [48]. We recall that δ is half the sum of positive roots: $\delta=\frac{1}{2}\sum_{\alpha\in R_+}\alpha$.

2.4 Asymptotic solutions. For generic λ operator $L=L(k)$ has $(n+1)!$ eigenfunctions with the eigenvalue $(\lambda,\lambda)-(\rho,\rho)$ with leading asymptotic $z^{w\lambda+\rho}$, correspondingly. Recall that $\rho=\frac{k}{2}\sum_{\alpha\in R_+}\alpha$. Asymptotic solutions are enumerated by the elements of the Weyl group $w\in W$. These solutions satisfy to the whole hypergeometric system of differential equations of Heckman and Opdam (Macdonald, Sekigushi, Debiard). Moreover, locally they provide a basis for all the solutions of the hypergeometric system cf. Corollary 3.11 ref. [45]. We denote asymptotic solution with leading asymptotic $z^{\lambda+\rho}$ by $\phi(\lambda+\rho(k),k,z)$. Asymptotic solutions are connected with many interesting parts of mathematics and physics, cf. refs. [23,36 ,37].

Theorem 2.5. *(Transformation law)(Opdam)*

$$\prod(z_i-z_j)^{2k-1}\prod z_i^{(1-2k)\frac{n}{2}}\phi(\lambda+\rho(k),k,z)=\phi(\lambda+\rho(1-k),1-k,z)$$

cf. Corollary 4.4 ref. [48].

The proposition is an easy corollary of the property 2.3. of operator L.

2.6 Note:. The homogeneity of

$$\prod(z_i-z_j)^{1-2k}\prod z_i^{(2k-1)\frac{n}{2}}$$

equals to zero.

Remark 2.7. The importance of the above simple theorem 2.5 is hardly possible to overvalue.

3. Integral representations

Consider the following variables: $z_l,\ l=1,...,n{+}1;\ t_{i,j},\ i=1,...,j,\ j=1,...,n$. It is convenient to organize these variables in the form of a pattern cf. fig. 1. The idea of such an organization is borrowed from

$$
\begin{array}{ccccccccc}
z_1 & & z_2 & & \dots & & \dots & & z_{n+1} \\
& t_{1,n} & & t_{2,n} & & \dots & & t_{n,n} & \\
& & \dots & & \dots & & \dots & & \\
& & & t_{1,2} & & t_{2,2} & & & \\
& & & & t_{1,1} & & & &
\end{array}
$$

FIGURE 1. Variables organized in a pattern

ref.[8], while variables t_{ij} itself have a nice geometric origin in elliptic coordinates cf. ref. [10]. Also these variables appear in Knizhnik-Zamolodchikov approach in ref. [38].

In ref. [43] we described contours for integration $\Delta_w = \Delta_w(z)$ which provide asymptotic solutions $\phi(w\lambda + \rho, k, z)$ for the Heckman-Opdam hypergeometric system of differential equations. We also obtained a multivalued form and made a natural convention about the phase of the form over cycle Δ_w. We assume that similar convention is made in theorems 3.2 and 4.1 below.

Theorem 3.1. *Let $w \in S_{n+1}$. Then for generic λ, k the integral of the multivalued form below over cycle Δ_w gives an asymptotic solution $\phi(w\lambda + \rho, k, z)$ to the Heckman-Opdam hypergeometric system of differential equations*

$$
\begin{aligned}
&\prod_{i=1}^{n+1} z_i^{\lambda_1 + \frac{kn}{2}} \prod_{i_1 > i_2} (z_{i_1} - z_{i_2})^{1-2k} \int_{\Delta_w(z)} \prod_{i,i_1} (z_i - t_{i_1,n})^{k-1} \\
&\times \prod_{j=1}^{n-1} \prod_{i,i_1} (t_{ij} - t_{i_1,j+1})^{k-1} \\
&\times \prod_{j=2}^{n} \prod_{i_1 > i_2} (t_{i_1,j} - t_{i_2,j})^{2-2k} \\
&\times \prod_{j=1}^{n} \prod_{i=1}^{j} t_{ij}^{\lambda_{n-j+2} - \lambda_{n-j+1} - k} \quad dt_{11} dt_{12} dt_{22} \dots dt_{nn} = a(w)\phi(w\lambda + \rho, k, z)
\end{aligned}
$$

$$\begin{aligned} a(w) =& e^{-2\pi i(\lambda,\delta)}e^{-\pi i(k-1)l(w)}(2i)^{\frac{n(n+1)}{2}}\Gamma(k)^{\frac{n(n+1)}{2}} \\ &\times \prod_{\alpha\in R_+}\frac{\Gamma((-w\lambda,\alpha^\vee))\sin(\pi(-w\lambda,\alpha^\vee))}{\Gamma((-w\lambda,\alpha^\vee)+k)} \end{aligned}$$

cf. theorems 6.1 and 6.3 of ref. [43].

Integral representation in theorem 3.1 has certain advantage. Namely, one can identify contour for integration for zonal spherical function itself cf. ref.[44].

Applying the transformation law to the integral representation of theorem 3.1. one obtains the following integral representation.

Theorem 3.2. *Let $w \in S_{n+1}$. Then for generic λ, k the integral of the multivalued form below over cycle Δ_w gives an asymptotic solution $\phi(w\lambda+\rho,k,z)$ to the Heckman-Opdam hypergeometric system of differential equations:*

$$\begin{aligned} \prod_{i=1}^{n+1} z_i^{\lambda_1+\frac{kn}{2}} \int_{\Delta_w(z)} & \prod_{i,i_1}(z_i - t_{i_1,n})^{-k} \\ & \times \prod_{j=1}^{n-1}\prod_{i,i_1}(t_{ij}-t_{i_1,j+1})^{-k} \\ & \times \prod_{j=2}^{n}\prod_{i_1>i_2}(t_{i_1,j}-t_{i_2,j})^{2k} \\ & \times \prod_{j=1}^{n}\prod_{i=1}^{j} t_{ij}^{\lambda_{n-j+2}-\lambda_{n-j+1}+k-1} \qquad dt_{11}dt_{12}dt_{22}\dots dt_{nn} = a(w)\phi(w\lambda+\rho,k,z) \end{aligned}$$

where

$$\begin{aligned} a(w) =& e^{-2\pi i(\lambda,\delta)}e^{\pi ikl(w)}(2i)^{\frac{n(n+1)}{2}}\Gamma(1-k)^{\frac{n(n+1)}{2}} \\ &\times \prod_{\alpha\in R_+}\frac{\Gamma((-w\lambda,\alpha^\vee))\sin(\pi(-w\lambda,\alpha^\vee))}{\Gamma((-w\lambda,\alpha^\vee)-k+1)} \end{aligned}$$

Remark 3.3. Calculation of leading asymptotic coefficient uses diagrams cf. ref. [43], section 1,2 and induction. Or can be obtained from theorem 3.1 simply by replacing k by $1-k$.

Remark 3.4. Compare integral representation in theorem 3.2 with the integral representation indicated in ref. [38], obtained with the help of symmetrization of solutions of trigonometric Knizhnik-Zamolodchikov equation. Also, cf. ref. [40] for the integral solutions of trigonometric Knizhnik-Zamolodchikov equations.

4. Generalized Selberg integral

Here is the main result of the paper.

Theorem 4.1. *(Generalized Selberg integral)*

$$\int_{\Delta_w(1)} \prod_{i_1=1}^{n} (1-t_{i_1,n})^{-(n+1)k}$$
$$\times \prod_{j=1}^{n-1} \prod_{i,i_1} (t_{ij}-t_{i_1,j+1})^{-k}$$
$$\times \prod_{j=2}^{n} \prod_{i_1>i_2} (t_{i_1,j}-t_{i_2,j})^{2k}$$
$$\times \prod_{j=1}^{n} \prod_{i=1}^{j} t_{ij}^{\lambda_{n-j+2}-\lambda_{n-j+1}+k-1} \quad dt_{11}dt_{12}dt_{22}\dots dt_{nn}$$
$$= (2\pi i)^{\frac{n(n+1)}{2}} e^{-2\pi i(\lambda,\delta)} e^{\pi i k l(w)} \Gamma(1-k)^{\frac{n(n+1)}{2}}$$
$$\times \prod_{\alpha\in R_+} \frac{1}{\Gamma((w\lambda,\alpha^\vee)-k+1)\Gamma((-w\lambda,\alpha^\vee)-k+1)}$$
$$\times \prod_{\alpha\in R_+} \frac{\Gamma((-\rho,\alpha^\vee)-k+1)}{\Gamma((-\rho,\alpha^\vee)+1)}$$

Remark 4.2. Remarkably the above constant does depend on $w \in S_{n+1}$ only in the phase factor $e^{\pi i k l(w)}$. Note also, that in ref. [43] we made a natural convention about the phase of the integrand ω_w over Δ_w.

Remark 4.3. The generalized Selberg integral can be conveniently rewritten as follows. Let's assign to each variable t_{ij} a simple root $\alpha(t_{ij})$ by the rule:

$$\alpha(t_{ij}) = \alpha(j) = \alpha_{n+1-j} = e_{n+1-j} - e_{n+2-j}$$

Note that to each variable of the same row we assign the same simple root. This assignment looks different from [38] only because we use different indexation of variables of integration t_{ij}.

Let also Λ_1 be the first fundamental weight.

Then one can rewrite the Selberg integral from theorem 4.1 as follows:

$$\begin{aligned}&\int \prod t_{ij}^{(\lambda-\rho,-\alpha(j))}\\ &\times \prod (1-t_{ij})^{k((n+1)\Lambda_1,-\alpha(j))}\\ &\times \prod (t_{ij}-t_{i'j'})^{k(-\alpha(j),-\alpha(j'))}\frac{dt_{11}}{t_{11}}\frac{dt_{12}}{t_{12}}\dots\frac{dt_{nn}}{t_{nn}}\end{aligned}$$

The theorem is an immediate application of the theorem 3.2 and the following theorem.

Theorem 4.4. *Evaluation theorem (Opdam).*

$$\phi(w\lambda+\rho(k),k,1)=\lim_{z\to 1}\phi(w\lambda+\rho(k),k,z)=\frac{\prod_{\alpha\in R_+}\frac{\Gamma((w\lambda,\alpha^\vee)+1)}{\Gamma((w\lambda,\alpha^\vee)-k+1)}}{\prod_{\alpha\in R_+}\frac{\Gamma(-(\rho,\alpha^\vee)+1)}{\Gamma(-(\rho,\alpha^\vee)-k+1)}}$$

cf. theorem 6.3 [48].

Recall that we restrict ourselves to the case of root system of type A_n.

Concluding remarks. In this paper we obtained a generalization of Selberg integral. Integrals of this type are important for the conformal field theory cf. ref. [2]. The Selberg integral considered in this paper can serve as an example of such integrals.

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